Exercise Problem Sets 8

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In the exercise section of Chapter 3, we first introduce the concepts of accumulation points, isolated points and derived set of a set as follows.

Definition 0.1. Let (M, d) be a normed vector space, and A be a subset of M.

1. A point $x \in M$ is called an ***accumulation point*** of A if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $A\backslash\{x\}$ such that $\{x_n\}_{n=1}^{\infty}$ converges to x.
2. A point $x \in A$ is called an ***isolated point*** (孤立點) (of A) if there exists no sequence in $A\backslash\{x\}$ that converges to x.
3. The ***derived set*** of A is the collection of all accumulation points of A, and is denoted by A'.

Problem 1. Let (M, d) be a metric space, and A be a subset of M.

1. Show that the collection of all isolated points of A is $A\backslash A'$.
2. Show that $A' = \bar{A}\backslash(A\backslash A')$. In other words, the derived set consists of all limit points that are not isolated points. Also show that $\bar{A}\backslash A' = A\backslash A'$.

Problem 2. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
2. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
3. $\mathrm{cl}(A \cap B) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(B)$. Find examples of that $\mathrm{cl}(A \cap B) \subsetneq \mathrm{cl}(A) \cap \mathrm{cl}(B)$.

Problem 3. Let A and B be subsets of a metric space (M, d). Show that

1. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.
2. $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.
3. $\mathrm{int}(A \cup B) \supseteq \mathrm{int}(A) \cup \mathrm{int}(B)$. Find examples of that $\mathrm{int}(A \cup B) \supsetneq \mathrm{int}(A) \cup \mathrm{int}(B)$.

Problem 4. Let (M, d) be a metric space, and A be a subset of M. Show that

$$\partial A = \big(A \cap \mathrm{cl}(M\backslash A)\big) \cup \big(\mathrm{cl}(A)\backslash A\big).$$

Problem 5. Recall that in a metric space (M, d), a subset A is said to be dense in S if subsets satisfy $A \subseteq S \subseteq \mathrm{cl}(A)$. For example, $\mathbb{Q}$ is dense in $\mathbb{R}$.

1. Show that if A is dense in S and if S is dense in T, then A is dense in T.
2. Show that if A is dense in S and $B \subseteq S$ is open, then $B \subseteq \mathrm{cl}(A \cap B)$.

Problem 6. Let A and B be subsets of a metric space (M, d). Show that

1. $\partial(\partial A) \subseteq \partial(A)$. Find examples of that $\partial(\partial A) \subsetneq \partial A$. Also show that $\partial(\partial A) = \partial A$ if A is closed.
2. $\partial(A \cup B) \subseteq \partial A \cup \partial B \subseteq \partial(A \cup B) \cup A \cup B$. Find examples of that equalities do not hold.
3. If $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$, then $\partial(A \cup B) = \partial A \cup \partial B$.
4. $\partial(A \cap B) \subseteq \partial A \cup \partial B$. Find examples of the equalities do not hold.
5. $\partial(\partial(\partial A)) = \partial(\partial A)$.

Problem 7. Let (M, d) be a metric space, and A be a subset of M. Show that $A \supseteq A'$ if and only if A is closed.

Problem 8. Show that the derived set of a set (in a metric space) is closed.

Problem 9. Let $A \subseteq \mathbb{R}^n$. Define the sequence of sets $A^{(m)}$ as follows: $A^{(0)} = A$ and $A^{(m+1)} =$ the derived set of $A^{(m)}$ for $m \in \mathbb{N}$. Complete the following.

1. Prove that each $A^{(m)}$ for $m \in \mathbb{N}$ is a closed set; thus $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$.
2. Show that if there exists some $m \in \mathbb{N}$ such that $A^{(m)}$ is a countable set, then A is countable.
3. For any given $m \in \mathbb{N}$, is there a set A such that $A^{(m)} \neq \varnothing$ but $A^{(m+1)} = \varnothing$?
4. Let A be uncountable. Then each $A^{(m)}$ is an uncountable set. Is it possible that $\bigcap_{m=1}^{\infty} A^{(m)} = \varnothing$?
5. Let $A = \left\{ \frac{1}{m} + \frac{1}{k} \,\middle|\, m - 1 > k(k-1), m, k \in \mathbb{N} \right\}$. Find $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$.

Problem 10. Recall that a cluster point x of a sequence $\{x_n\}_{n=1}^{\infty}$ satisfies that

$$\forall\, \varepsilon > 0, \#\left\{ n \in \mathbb{N} \,\middle|\, x_n \in B(x, \varepsilon) \right\} = \infty\,.$$

Show that the collection of cluster points of a sequence (in a metric space) is closed.

Problem 11. Let (M, d) be a metric space, and $A \subseteq M$. Show that A is dense in M if and only if $A \cap U \neq \varnothing$ for all non-empty open set $U \subseteq M$.

Problem 12. Determine whether the following sets are open or not.

1. $\bigcup_{n=1}^{\infty} [-2 + \frac{1}{n}, 2 + \frac{1}{n}]$.
2. $\bigcup_{n=1}^{\infty} [-2 - \frac{1}{n}, 2 - \frac{1}{n}]$.
3. $\bigcup_{n=1}^{\infty} [-2 + \frac{1}{n}, 2 - \frac{1}{n}]$.
4. $\bigcup_{n=1}^{\infty} [-2 - \frac{1}{n}, 2 + \frac{1}{n}]$.

Problem 13. Let $(\mathcal{V}, \|\cdot\|)$ ba a normed vector space, and C be a non-empty convex set in $\mathcal{V}$.

1. Show that $\bar{C}$ is convex.

2. Show that if $\boldsymbol{x} \in \mathring{C}$ and $\boldsymbol{y} \in \bar{C}$, then $(1-\lambda)\boldsymbol{x}+\lambda\boldsymbol{y} \in \mathring{C}$ for all $\lambda \in (0,1)$. This result is sometimes called the ***line segment principle***.

3. Show that $\mathring{C}$ is convex (you may need the conclusion in 2 to prove this).

4. Show that $\mathrm{cl}(\mathring{C}) = \mathrm{cl}(C)$.

5. Show that $\mathrm{int}(\bar{C}) = \mathrm{int}(C)$.

Hint: 2. Prove by contradiction.
3 and 4. Use the line segment principle.
5. Show that $\boldsymbol{x} \in \mathrm{int}(\bar{C})$ can be written as $(1-\lambda)\boldsymbol{y} + \lambda\boldsymbol{z}$ for some $\boldsymbol{y} \in \mathring{C}$ and $\boldsymbol{z} \in B(\boldsymbol{x}, \varepsilon) \subseteq \bar{C}$.

Problem 14. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space with induced norm $\|\cdot\|$, and C be a non-empty convex set in $\mathcal{V}$. Show that for each $\boldsymbol{x} \in \mathcal{V}$, there is a unique point $\boldsymbol{x}_{_C} \in \bar{C}$ such that

$$\|\boldsymbol{x} - \boldsymbol{x}_{_C}\| = \mathrm{dist}(\boldsymbol{x}, C) = \inf\big\{\|\boldsymbol{x} - \boldsymbol{y}\| \,\big|\, \boldsymbol{y} \in C\big\}.$$

How about if $(\mathcal{V}, \|\cdot\|)$ is a norm vector space whose norm $\|\cdot\|$ is not induced from an inner product?

Problem 15. In this problem we discuss the concept of supporting hyperplanes in an inner product spaces. Recall that a hyperplane in an n-dimensional vector space is an $(n-1)$-dimensional subspace.

Definition 0.2. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space, and S be a subset of $\mathcal{V}$. A hyperplane H is called a supporting hyperplane of S if

1. S is a subset of one of the two closed half-spaces bounded by H;

2. H contains at least one boundary point of S.

If $\boldsymbol{x} \in \partial S$, a supporting hyperplane of S at $\boldsymbol{x}$ is a supporting hyperplane of S containing $\boldsymbol{x}$.

In particular, if the norm is induced by an inner product $\langle\cdot,\cdot\rangle$, H is a supporting hyperplane of S at $\boldsymbol{x} \in \partial S$ if there exists a unit vector $\boldsymbol{n} \in \mathcal{V}$ such that $H = \big\{\boldsymbol{y} \big| \langle \boldsymbol{n}, \boldsymbol{y} - \boldsymbol{x}\rangle = 0\big\}$ and $S \subseteq \big\{\boldsymbol{y} \big| \langle \boldsymbol{n}, \boldsymbol{y} - \boldsymbol{x}\rangle \leqslant 0\big\}$.

Let $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ ba an inner product space, and C be a non-empty convex set in $\mathcal{V}$. Show that for each $\boldsymbol{x} \in \partial C$, there is a supporting hyperplane of C at $\boldsymbol{x}$.

Problem 16. Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. Show that for all $\boldsymbol{x} \in \mathcal{V}$ and $r > 0$,

$$\mathrm{int}\big(B[\boldsymbol{x}, r]\big) = B(\boldsymbol{x}, r)\,.$$

Is the identity above true in general metric space?

Problem 17. Let $\mathcal{M}_{n\times n}$ denote the collection of all $n \times n$ square real matrices, and $(\mathcal{M}_{n\times n}, \|\cdot\|_{p,q})$ be a normed space with norm $\|\cdot\|_{p,q}$ given in Problem **??**. Show that the set

$$\mathrm{GL}(n) \equiv \big\{A \in \mathcal{M}_{n\times n} \,\big|\, \det(A) \neq 0\big\}$$

is an open set in $\mathcal{M}_{n\times n}$. The set $\mathrm{GL}(n)$ is called the general linear group.

Problem 18. Show that every open set in $\mathbb{R}$ is the union of at most countable collection of disjoint open intervals; that is, if $U \subseteq \mathbb{R}$ is open, then

$$U = \bigcup_{k \in \mathcal{I}} (a_k, b_k)\,,$$

where $\mathcal{I}$ is countable, and $(a_k, b_k) \cap (a_\ell, b_\ell) = \varnothing$ if $k \neq \ell$.
Hint: For each point $x \in U$, define $L_x = \big\{y \in \mathbb{R} \,\big|\, (y, x) \subseteq U\big\}$ and $R_x = \big\{y \in \mathbb{R} \,\big|\, (x, y) \subseteq U\big\}$. Define $I_x = (\inf L_x, \sup R_x)$. Show that $I_x = I_y$ if $(x, y) \in U$ and if $(x, y) \nsubseteq U$ then $I_x \cap I_y = \varnothing$

Problem 19. Let (M, d) be a metric space. A set $A \subseteq M$ is said to be ***perfect*** if $A = A'$ (so that there is no isolated points). The Cantor set is constructed by the following procedure: let $E_0 = [0, 1]$. Remove the segment $\big(\frac{1}{3}, \frac{2}{3}\big)$, and let E_1 be the union of the intervals

$$\big[0, \tfrac{1}{3}\big], \big[\tfrac{2}{3}, 1\big].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$\big[0, \tfrac{1}{9}\big], \big[\tfrac{2}{9}, \tfrac{3}{9}\big], \big[\tfrac{6}{9}, \tfrac{7}{9}\big], \big[\tfrac{8}{9}, 1\big].$$

Continuing in this way, we obtain a sequence of closed set E_k such that

(a) $E_1 \supseteq E_2 \supseteq E_2 \supseteq \cdots$;

(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set $C = \bigcap_{n=1}^{\infty} E_n$ is called the ***Cantor set***.

1. Show that C is a perfect set.
2. Show that C is uncountable.
3. Find $\operatorname{int}(C)$.

Problem 20. In this problem you are asked to prove the Baire Category Theorem.

Theorem (Baire Category Theorem). *Let (X, d) be a complete metric space. If $U_1, U_2, \ldots$ are dense open subsets of X, then*

$$\bigcap_{n=1}^{\infty} U_n$$

is dense in X.

Complete the following.

1. By Problem 11, it suffices to show that

$$W \cap \bigcap_{n=1}^{\infty} U_n \neq \varnothing$$

for every nonempty open set $W \subseteq X$. Let W be a nonempty open set. Use the fact that each U_n is dense and open to recursively construct closed balls

$$B[x_1, r_1] \supseteq B[x_2, r_2] \supseteq \cdots$$

such that

$$B[x_n, r_n] \subseteq U_n \cap B(x_{n-1}, r_{n-1}) \qquad \text{and} \qquad r_n < 2^{-n}.$$

2. Show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and hence converges to some limit $x \in X$.

3. Show that

$$x \in \bigcap_{n=1}^{\infty} B[x_n, r_n],$$

and that x is the only point in this intersection.

4. Deduce that

$$x \in W \cap \bigcap_{n=1}^{\infty} U_n,$$

and therefore

$$W \cap \bigcap_{n=1}^{\infty} U_n \neq \varnothing.$$

Conclude that $\bigcap_{n=1}^{\infty} U_n$ is dense in X.