Exercise Problem Sets 9

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Problem 1. Let $\mathcal{V}$ be a vector fields over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \subseteq \mathcal{V}$ is a basis for $\mathcal{V}$; that is, every $\boldsymbol{x} \in \mathcal{V}$ can be uniquely expressed as

$$\boldsymbol{x} = x^{(1)}\mathbf{e}_1 + x^{(2)}\mathbf{e}_2 + \cdots + x^{(n)}\mathbf{e}_n = \sum_{i=1}^{n} x^{(i)}\mathbf{e}_i \,.$$

Define $\|\boldsymbol{x}\|_2 = \Big(\sum\limits_{i=1}^{n} \big|x^{(i)}\big|^2 \Big)^{\frac{1}{2}}$.

1. Show that $\|\cdot\|_2$ is a norm on $\mathcal{V}$.
2. Show that K is compact in $(\mathcal{V}, \|\cdot\|_2)$ if and only if K is closed and bounded.

Problem 2. Let (M, d) be a metric space.

1. Show that a closed subset of a compact set is compact.
2. Show that the union of a finite number of sequentially compact subsets of M is compact.
3. Show that the intersection of an arbitrary collection of sequentially compact subsets of M is sequentially compact.

Problem 3. A metric space (M, d) is said to be ***separable*** if there is a countable subset A which is dense in M. Show that every sequentially compact set is separable.
Hint: Consider the total boundedness using balls with radius $\dfrac{1}{n}$ for $n \in \mathbb{N}$.

Problem 4. Given $\{a_k\}_{k=1}^{\infty} \subseteq \mathbb{R}$ a bounded sequence, define

$$A = \Big\{ x \in \mathbb{R} \,\Big|\, \text{there exists a subsequence } \big\{a_{k_j}\big\}_{j=1}^{\infty} \text{ such that } \lim_{j\to\infty} a_{k_j} = x \Big\} \,.$$

Show that A is a non-empty sequentially compact set in $\mathbb{R}$. Furthermore , $\limsup\limits_{k\to\infty} a_k = \sup A$ and $\liminf\limits_{k\to\infty} a_k = \inf A$.

Problem 5. Let (M, d) be a metric space.

1. Show that if M is complete and A is a totally bounded subset of M, then $\mathrm{cl}(A)$ is sequentially compact.
2. Show that M is complete if and only if every totally bounded sequence has a convergent subsequence.

Problem 6. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} |x_1 - y_1| & \text{if } x_2 = y_2 \,, \\ |x_1 - y_1| + |x_2 - y_2| + 1 & \text{if } x_2 \neq y_2 \,. \end{cases} \quad \text{where } x = (x_1, x_2) \text{ and } y = (y_1, y_2).$$

Problem 6 of Exercise 6 shows that d is a metric on $\mathbb{R}^2$. Consider the metric space $(\mathbb{R}^2, d)$.

1. Find $B(x,r)$ with $r<1$, $r=1$ and $r>1$.

2. Show that the set $\{c\}\times[a,b]\subseteq(\mathbb{R}^2,d)$ is closed and bounded.

3. Examine whether the set $\{c\}\times[a,b]\subseteq(\mathbb{R}^2,d)$ is sequentially compact or not.

Problem 7. Let $\{x_k\}_{k=1}^{\infty}$ be a convergent sequence in a metric space, and $x_k\to x$ as $k\to\infty$. Show that the set $A\equiv\{x_1,x_2,\cdots,\}\cup\{x\}$ is sequentially compact.

Problem 8. 1. Show the so-called "***Finite Intersection Property***":

> Let (M,d) be a metric space, and K be a subset of M. Then K is compact if and if for any family of closed subsets $\{F_\alpha\}_{\alpha\in I}$, we have
> $$K\cap\bigcap_{\alpha\in I}F_\alpha\neq\varnothing$$
> whenever $K\cap\bigcap_{\alpha\in J}F_\alpha\neq\varnothing$ for all $J\subseteq I$ satisfying $\#J<\infty$.

2. Show the so-called "***Nested Set Properpty***":

> Let (M,d) be a metric space. If $\{K_n\}_{n=1}^{\infty}$ is a sequence of non-empty compact sets in M such that $K_j\supseteq K_{j+1}$ for all $j\in\mathbb{N}$, then there exists at least one point in $\bigcap_{j=1}^{\infty}K_j$; that is,
> $$\bigcap_{j=1}^{\infty}K_j\neq\varnothing\,.$$

Problem 9. Let (M,d) be a metric space, and M itself is a sequentially compact set. Show that if $\{F_k\}_{k=1}^{\infty}$ is a sequence of closed sets such that $\operatorname{int}(F_k)=\varnothing$, then $M\setminus\bigcup_{k=1}^{\infty}F_k\neq\varnothing$.

Problem 10. Let ℓ^2 be the collection of all sequences $\{x_k\}_{k=1}^{\infty}\subseteq\mathbb{R}$ such that $\sum_{k=1}^{\infty}|x_k|^2<\infty$. In other words,
$$\ell^2=\Big\{\{x_k\}_{k=1}^{\infty}\,\Big|\,x_k\in\mathbb{R}\text{ for all }k\in\mathbb{N},\ \sum_{k=1}^{\infty}|x_k|^2<\infty\Big\}\,.$$
Define $\|\cdot\|_2:\ell^2\to\mathbb{R}$ by
$$\big\|\{x_k\}_{k=1}^{\infty}\big\|_2=\Big(\sum_{k=1}^{\infty}|x_k|^2\Big)^{\frac{1}{2}}\,.$$

1. Show that $\|\cdot\|_2$ is a norm on ℓ^2. The normed space $(\ell^2,\|\cdot\|)$ usually is denoted by ℓ^2.

2. Show that $\|\cdot\|_2$ is induced by an inner product.

3. Show that $(\ell^2,\|\cdot\|_2)$ is complete.

4. Let $A=\big\{\boldsymbol{x}\in\ell^2\,\big|\,\|\boldsymbol{x}\|_2\leqslant 1\big\}$. Is A sequentially compact or not?

Problem 11. Let A, B be two non-empty subsets in $\mathbb{R}^n$. Define

$$d(A,B) = \inf\left\{\|x-y\|_2 \,\middle|\, x \in A, y \in B\right\}$$

to be the distance between A and B. When $A = \{x\}$ is a point, we write $d(A,B)$ as $d(x,B)$ (which is consistent with the one given in Proposition **??**).

(1) Prove that $d(A,B) = \inf\left\{d(x,B) \,\middle|\, x \in A\right\}$.

(2) Show that $\left|d(x_1,B) - d(x_2,B)\right| \leqslant \|x_1 - x_2\|_2$ for all $x_1, x_2 \in \mathbb{R}^n$.

(3) Define $B_\varepsilon = \left\{x \in \mathbb{R}^n \,\middle|\, d(x,B) < \varepsilon\right\}$ be the collection of all points whose distance from B is less than ε. Show that B_ε is open and $\bigcap_{\varepsilon>0} B_\varepsilon = \mathrm{cl}(B)$.

(4) If A is sequentially compact, show that there exists $x \in A$ such that $d(A,B) = d(x,B)$.

(5) If A is closed and B is sequentially compact, show that there exists $x \in A$ and $y \in B$ such that $d(A,B) = d(x,y)$.

(6) If A and B are both closed, does the conclusion of (5) hold?

Problem 12. Let $\mathcal{K}(n)$ denote the collection of all non-empty sequentially compact sets in $\mathbb{R}^n$. Define the Hausdorff distance of $K_1, K_2 \in \mathcal{K}(n)$ by

$$d^H(K_1,K_2) = \max\left\{\sup_{x\in K_2} d(x,K_1), \sup_{x\in K_1} d(x,K_2)\right\},$$

in which $d(x,K)$ is the distance between x and K given in Problem 11. Show that $(\mathcal{K}(n), d^H)$ is a metric space.

Problem 13. Let $M = \left\{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 \leqslant 1\right\}$ with the standard metric $\|\cdot\|_2$. Show that $A \subseteq M$ is sequentially compact if and only if A is closed.

Problem 14. 1. Let $\{x_k\}_{k=1}^\infty \subseteq \mathbb{R}$ be a sequence in $(\mathbb{R}, |\cdot|)$ that converges to x and let $A_k = \{x_k, x_{k+1}, \cdots\}$. Show that $\{x\} = \bigcap_{k=1}^\infty \overline{A_k}$. Is this true in any metric space?

2. Suppose that $\{K_j\}_{j=1}^\infty$ is a sequence of comapct non-empty sets satisfying the nested set property; that is, $K_j \supseteq K_{j+1}$, and $\mathrm{diam}(K_j) \to 0$ as $j \to \infty$, where

$$\mathrm{diam}(K_j) = \sup\left\{d(x,y) \,\middle|\, x, y \in K_j\right\}.$$

Show that there is exactly one point in $\bigcap_{j=1}^\infty K_j$.

Problem 15. Let (M,d) be a metric space, and A be a subset of M satisfying that every sequence in A has a convergent subsequence (with limit in M). Show that A is pre-compact.
Remark: Together with Remark 3.61 in the Lecture Note, we conclude that a subset A is pre-compact if and only if A has the property that "every sequence in A has a convergent subsequence".