Exercise Problem Sets 10

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Problem 1. Use whatever methods you know to find the following limits:

1. $\lim_{x\to 0^+}(1+\sin 2x)^{\frac{1}{x}}$;
2. $\lim_{x\to -\infty}\left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)$;
3. $\lim_{x\to 1}(2-x)^{\sec\frac{\pi x}{2}}$;
4. $\lim_{x\to\infty} x\left(\frac{\pi}{2}-\sin^{-1}\frac{x}{\sqrt{x^2+1}}\right)$;
5. $\lim_{x\to\infty} x\left(e^{-1}-\left(\frac{x}{x+1}\right)^x\right)$;
6. $\lim_{x\to\infty}\left(\frac{a^x-1}{x(a-1)}\right)^{\frac{1}{x}}$, where $a>0$ and $a\neq 1$.

Problem 2. In this problem we prove the Intermediate Value Theorem:

> Let $f:[a,b]\to\mathbb{R}$ be continuous (at every point of $[a,b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big) \quad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty}\subseteq[a,b].$$
>
> If $f(a)f(b)<0$, then there exists $c\in[a,b]$ such that $f(c)=0$.

Complete the following.

1. W.L.O.G, we can assume that $f(a)<0$. Define the set $S=\{x\in[a,b]\,|\,f(x)>0\}$. Show that $\inf S$ exists.
2. Let $c=\inf S$. Show that $f(c)\geqslant 0$.
3. Conclude that $f(c)\leqslant 0$ as well.

Hint:

1. For 2, show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in S such that $c_n\to c$ as $n\to\infty$.
2. For 3, show that there exists a sequence $\{c_n\}_{n=1}^{\infty}$ in $[a,c)$ such that $c_n\to c$ as $n\to\infty$.

Problem 3. In this problem we prove the Extreme Value Theorem:

> Let $f:[a,b]\to\mathbb{R}$ be continuous (at every point of $[a,b]$); that is,
>
> $$\lim_{n\to\infty} f(x_n) = f\Big(\lim_{n\to\infty} x_n\Big) \quad \text{for all convergent sequence } \{x_n\}_{n=1}^{\infty}\subseteq[a,b].$$
>
> Then there exist $c,d\in[a,b]$ such that $f(c)=\sup_{x\in[a,b]} f(x)$ and $f(d)=\inf_{x\in[a,b]} f(x)$.

Complete the following.

1. Show that there exist sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ in $[a,b]$ such that

$$\lim_{n\to\infty} f(c_n)=\sup_{x\in[a,b]} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f(d_n)=\inf_{x\in[a,b]} f(x)\,.$$

2. Apply the Bolzano-Weierstrass Property of $\mathbb{R}$ to extract convergent subsequences $\{c_{n_k}\}_{k=1}^{\infty}$ and $\{d_{n_k}\}_{k=1}^{\infty}$ with limit c and d, respectively. Show that $c, d \in [a, b]$.

3. Show that $f(c) = \sup_{x\in[a,b]} f(x)$ and $f(d) = \inf_{x\in[a,b]} f(x)$.

Problem 4. Complete the following.

1. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{x\to 0}\lim_{y\to 0} f(x, y) \quad \text{and} \quad \lim_{y\to 0}\lim_{x\to 0} f(x, y)$$

exist but are not equal.

2. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the two limits above exist and are equal but f is not continuous.

3. Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through the origin but is not continuous.

Problem 5. Complete the following.

1. Show that the projection map $f : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x \end{array}$ is continuous.

2. Show that if $U \subseteq \mathbb{R}$ is open, then $A = \left\{(x, y) \in \mathbb{R}^2 \,\middle|\, x \in U\right\}$ is open.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $U \subseteq \mathbb{R}$ such that $f(U)$ is not open.

Problem 6. Show that $f : A \to \mathbb{R}^m$, where $A \subseteq \mathbb{R}^n$, is continuous if and only if for every $B \subseteq A$,

$$f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B)) .$$

Problem 7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfy $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$.

1. Show that $T(rx) = rT(x)$ for all $r \in \mathbb{Q}$ and $x \in \mathbb{R}^n$.

2. Suppose that T is continuous on $\mathbb{R}^n$. Show that T is linear; that is, $T(cx + y) = cT(x) + T(y)$ for all $c \in \mathbb{R}$, $x, y \in \mathbb{R}^n$.

3. Suppose that T is continuous at some point x_0 in $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

4. Suppose that T is bounded on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

5. Suppose that T is bounded from above (or below) on some open subset of $\mathbb{R}^n$. Show that T is continuous on $\mathbb{R}^n$.

6. Construct a $T : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every point of $\mathbb{R}$, but $T(x+y) = T(x)+T(y)$ for all $x, y \in \mathbb{R}$.

Problem 8. Let C be a convex set in $\mathbb{R}^n$ equipped with inner product $\langle \cdot, \cdot \rangle$ (for the definition of convex sets, see Problem 10 of Exercise), and $f : C \to \mathbb{R}$ be a convex function; that is,

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) \qquad \forall\, t \in [0,1] \text{ and } x, y \in C\,.$$

Complete the following.

1. Show that f is locally bounded (from above); that is, for every $x_0 \in \operatorname{int}(C)$ there exists $\delta > 0$ such that f is bounded (from above) on $B(x_0, \delta)$.

2. Show that f is locally Lipschitz; that is, for every $x_0 \in \operatorname{int}(C)$ there exists $\delta > 0$ and $M = M_\delta > 0$ such that

   $$\big|f(x) - f(y)\big| \leqslant M_\delta \|x - y\| \qquad \forall\, x, y \in B(x_0, \delta)\,.$$

   In particular, a convex function (defined on a subset of $\mathbb{R}^n$) is continuous in the interior of its domain.

3. For $x_0 \in \operatorname{int}(C)$, define the sub-differential

   $$\partial f(x_0) = \Big\{ p \in \mathbb{R}^n \,\Big|\, f(x) \geqslant f(x_0) + \langle p, x - x_0 \rangle \text{ for all } x \in C \Big\}\,.$$

   Show that $\partial f(x_0)$ is non-empty, convex and compact.

4. Show that f attains a local minimum at $x_0 \in \operatorname{int}(C)$ if and only if $0 \in \partial f(x_0)$.

Problem 9. Let (M, d) be a metric space, $A \subseteq M$, and $f : A \to \mathbb{R}$. For $a \in A'$, define

$$\begin{aligned}
\liminf_{x \to a} f(x) &= \lim_{r \to 0^+} \inf \big\{ f(x) \,\big|\, x \in B(a, r) \cap A \backslash \{a\} \big\}\,, \\
\limsup_{x \to a} f(x) &= \lim_{r \to 0^+} \sup \big\{ f(x) \,\big|\, x \in B(a, r) \cap A \backslash \{a\} \big\}\,.
\end{aligned}$$

Complete the following.

1. Show that both $\liminf\limits_{x \to a} f(x)$ and $\limsup\limits_{x \to a} f(x)$ exist (which may be $\pm\infty$), and

   $$\liminf_{x \to a} f(x) \leqslant \limsup_{x \to a} f(x)\,.$$

   Furthermore, there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty \subseteq A \backslash \{a\}$ such that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ both converge to a, and

   $$\lim_{n \to \infty} f(x_n) = \liminf_{x \to a} f(x) \qquad \text{and} \qquad \lim_{n \to \infty} f(y_n) = \limsup_{x \to a} f(x)\,.$$

2. Let $\{x_n\}_{n=1}^\infty \subseteq A \backslash \{a\}$ be a convergent sequence with limit a. Show that

   $$\liminf_{x \to a} f(x) \leqslant \liminf_{n \to \infty} f(x_n) \leqslant \limsup_{n \to \infty} f(y_n) \leqslant \limsup_{x \to a} f(x)\,.$$

3. Show that $\lim\limits_{x \to a} f(x) = \ell$ if and only if

   $$\liminf_{x \to a} f(x) = \limsup_{x \to a} f(x) = \ell\,.$$

4. Show that $\liminf\limits_{x\to a} f(x) = \ell \in \mathbb{R}$ if and only if the following two conditions hold:

   (a) for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\ell - \varepsilon < f(x)$ for all $x \in B(a,\delta) \cap A\backslash\{a\}$;

   (b) for all $\varepsilon > 0$ and $\delta > 0$, there exists $x \in B(a,\delta) \cap A\backslash\{a\}$ such that $f(x) < \ell + \varepsilon$.

   Formulate a similar criterion for limsup and for the case that $\ell = \pm\infty$.

5. Compute the liminf and limsup of the following functions at any point of $\mathbb{R}$.

   (a) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^{\complement}, \\ \dfrac{1}{p} & \text{if } x = \dfrac{q}{p} \text{ with } (p,q) = 1,\ q > 0,\ p \neq 0. \end{cases}$

   (b) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{Q}^{\complement}. \end{cases}$

Problem 10. Let (M,d) be a metric space, and $A \subseteq M$. A function $f : A \to \mathbb{R}$ is called ***lower semi-continuous*** / ***upper semi-continuous*** at $a \in A$ if either $a \in A\backslash A'$ or $\liminf\limits_{x\to a} f(x) \geqslant f(a)$, / $\limsup\limits_{x\to a} f(x) \leqslant f(a)$, and is called lower/upper semi-continuous on A if f is lower/uppser semi-continuous at a for all $a \in A$.

1. Show that $f : A \to \mathbb{R}$ is lower semi-continuous on A if and only if $f^{-1}((-\infty, r])$ is closed relative to A. Also show that $f : A \to \mathbb{R}$ is upper semi-continuous on A if and only if $f^{-1}([r,\infty))$ is closed relative to A.

2. Show that f is lower semi-continuous on A if and only if for all convergent sequences $\{x_n\}_{n=1}^{\infty} \subseteq A$ and $\{s_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ satisfying $f(x_n) \leqslant s_n$ for all $n \in \mathbb{N}$, we have

$$f\big(\lim_{n\to\infty} x_n\big) \leqslant \lim_{n\to\infty} s_n .$$

3. Let $\{f_\alpha\}_{\alpha\in I}$ be a family of lower semi-continuous functions on A. Prove that $f(x) = \sup\limits_{\alpha\in I} f_\alpha(x)$ is lower semi-continuous on A.

4. Let A be a perfect set (that is, A contains no isolated points) and $f : A \to \mathbb{R}$ be given. Define

$$f^*(x) = \limsup_{y\to x} f(y) \qquad \text{and} \qquad f_*(x) = \liminf_{y\to x} f(y).$$

   Show that f^* is upper semi-continuous and f_* is lower semi-continuous, and $f_*(x) \leqslant f(x) \leqslant f^*(x)$ for all $x \in A$. Moreover, if g is a lower semi-continuous function on A such that $g(x) \leqslant f(x)$ for all $x \in A$, then $g \leqslant f_*$.