Exercise Problem Sets 10

Nov. 13. 2026

Problem 1. Show that the following functions (defined by integrals) are one-to-one and find $(f^{-1})'(0)$.

1. $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\,dt.$ 2. $f(x) = \displaystyle\int_2^x \frac{dt}{\sqrt{1+t^4}}.$

Problem 2. Let f be an one-to-one, twice differentiable function with an inverse function g.

1. Show that g is twice differentiable function and find g''.
2. Show that if in addition f is strictly increasing and the graph of f is concave upward, then the graph of g is concave downward.

Problem 3. Compute $\displaystyle\int_{-\sqrt{\frac{3}{2}}}^{1} \frac{dx}{\sqrt{2-x^2}}$ using the following substitution of variables:

1. $x = \sqrt{2}\sin t.$ 2. $x = -\sqrt{2}\sin t.$ 3. $x = \sqrt{2}\cos t.$ 4. $x = -\sqrt{2}\cos t.$

Problem 4. Compute the indefinite integral $\displaystyle\int \sec^3 x\,dx$ using the substitution $t = \sec x - \tan x$.

Hint: Note that $\dfrac{1}{t} = \sec x + \tan x$.

Problem 5. Use the substitution of variable $t = \tan\dfrac{x}{2}$ to find the integral $\displaystyle\int \csc x\,dx$.

Problem 6. Find the following indefinite integrals.

1. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx.$ 2. $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx.$ 3. $\displaystyle\int \frac{dx}{x(\ln x^2)^3}.$ 4. $\displaystyle\int \frac{(1+\ln x)^2}{x}\,dx.$
5. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx.$ 6. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx.$

Problem 7. Show that $\ln x \leqslant \dfrac{x-1/x}{2}$ for all $0 < x \leqslant 1$ and $\ln x \geqslant \dfrac{x-1/x}{2}$ for all $x \geqslant 1$.

Problem 8. Show that $\dfrac{2x}{2+x} < \ln(1+x) < \dfrac{x(2+x)}{2(1+x)}$ for all $x > 0$.

Problem 9. Show that for all natural number n,

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}x^k}{k} \leqslant \ln(1+x) \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}x^k}{k} \qquad \forall\, x > 0\,.$$

Problem 10. Find the derivative of the following functions by first taking the logarithm (base e) and then differentiating.

1. $y = \dfrac{x(x-1)^{\frac{3}{2}}}{\sqrt{x+1}}$, $x > 1.$ 2. $y = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}$, $x > 2$

Problem 11. Show that $\ln\left(\dfrac{x+y}{2}\right) \geqslant \dfrac{\ln x + \ln y}{2}$ for all $x, y > 0$.

Problem 12. Show that $\dfrac{1}{y} < \dfrac{\ln x - \ln y}{x - y} < \dfrac{1}{x}$ for all $0 < x < y$.

Problem 13. Show that $\dfrac{2}{x+y} < \dfrac{\ln x - \ln y}{x - y} < \dfrac{x+y}{2xy}$ for all $0 < x < y$.

Problem 14. Show that $\sqrt{xy} < \dfrac{x - y}{\ln x - \ln y} < \dfrac{x+y}{2}$ for all $0 < x < y$. The quantity in the middle of the inequality is called the logarithmic mean.