Exercise Problem Sets 11

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Problem 1. Show that the following functions are decreasing on $(0, \infty)$.

1. $y = \big(1 + \dfrac{1}{2x}\big)^{x+0.5}$. 2. $y = \big(1 + \dfrac{1}{x}\big)^{x+0.5}$.

Problem 2. In this example you are asked to compute the integral of $y = xe^x$ by the Riemann sum. Complete the following.

1. Show that if $r \neq 1$, then $\displaystyle\sum_{k=1}^{n} kr^k = \frac{r(1-r^n)}{(1-r)^2} - \frac{nr^{n+1}}{1-r}$.
2. Compute $\displaystyle\int_0^a xe^x\,dx$ by the limit the Riemann sum of $y = xe^x$ for regular partition using the right end-point rule.
3. Find an anti-derivative of $y = xe^x$.

Problem 3 (Integrating Factor).

1. Let $f, g : [a, b] \to \mathbb{R}$ be a continuous function, F be an anti-derivative of f, and $y : [a, b] \to \mathbb{R}$ satisfies that
$$y' + f(x)y = g(x)\,. \qquad (\star)$$
Find an expression of y.
2. Find the function y satisfying $y' + x^2 y = 2x^2$ and $y(0) = 1$.

Hint: Multiply both sides of $(\star)$ by $\exp(F(x))$ and observe that the left-hand side is the derivative of a certain function.

Problem 4. Prove the following inequalities.

1. $e^x > 1 + \ln(1 + x)$ for all $x > 0$.
2. $e^x > 1 + (1 + x)\ln(1 + x)$ for all $x > 0$.
3. $e^x \geqslant 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$ for all $x \geqslant 0$ and $n \in \mathbb{N}$.

Problem 5. Let a, b be two positive numbers, p, q any nonzero numbers, and $p < q$. Prove that
$$\big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}} \leqslant \big[\theta a^q + (1-\theta)b^q\big]^{\frac{1}{q}} \qquad \forall\, \theta \in (0, 1)\,.$$

Hint: Show that the function $f(p) = \big[\theta a^p + (1-\theta)b^p\big]^{\frac{1}{p}}$ is an increasing function of p.

Problem 6. 1. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.

2. Show that $\ln x < \dfrac{x}{e}$ for all $x \neq e$.

3. Show that $x^e < e^x$ for all $x \neq e$.

4. Show that if $e \leqslant A < B$, then $A^B > B^A$.

Problem 7 (Implicit Differentiation).

1. Find y' if $e^{\frac{x}{y}} = x - y$.

2. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

3. Find an equation of the tangent line to the curve $1 + \ln xy = e^{x-y}$ at the point $(1, 1)$

Problem 8. Evaluate the following limits. Use L'Hôpital's Rule where appropriate. If L'Hôpital's Rule does not apply, explain why.

1. $\lim\limits_{x\to 0^+} \dfrac{\arctan(2x)}{\ln x}$. 2. $\lim\limits_{x\to 0^+} \dfrac{x^x - 1}{\ln x + x - 1}$. 3. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{\cos x + e^x - 1}$.

4. $\lim\limits_{x\to 0} \dfrac{x^a - 1}{x^b - 1}$, where $b \neq 0$. 5. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$. 6. $\lim\limits_{x\to a^+} \dfrac{\cos x \cdot \ln(x-a)}{\ln(e^x - e^a)}$.

7. $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\arctan x}\right)$. 8. $\lim\limits_{x\to\infty} (x - \ln x)$. 9. $\lim\limits_{x\to 1^+} \ln(x^7 - 1) - \ln(x^5 - 1)$.

10. $\lim\limits_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$. 11. $\lim\limits_{x\to\infty} x^{e^{-x}}$. 12. $\lim\limits_{x\to 1} (2-x)^{\tan(\pi x/2)}$. 13. $\lim\limits_{x\to 0^+} (\sin x)(\ln x)$.

Problem 9. Evaluate the following limits:

1. $\lim\limits_{x\to\infty} x\left[\left(1 + \dfrac{1}{x}\right)^x - e\right]$. 2. $\lim\limits_{x\to\infty} \left\{\dfrac{e}{2}x + x^2\left[\left(1 + \dfrac{1}{x}\right)^x - e\right]\right\}$.

3. $\lim\limits_{x\to\infty} x\left[\left(1 + \dfrac{1}{x}\right)^x - e\ln\left(1 + \dfrac{1}{x}\right)^x\right]$. 4. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$. 5. $\lim\limits_{x\to\infty} \left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

6. $\lim\limits_{x\to\infty} \left(x - x^2 \ln\dfrac{1+x}{x}\right)$. 7. $\lim\limits_{x\to\infty} \left[\dfrac{1}{x} \cdot \dfrac{a^x - 1}{a - 1}\right]^{\frac{1}{x}}$, where $a > 0$ and $a \neq 1$.

Remark 0.1. In some of the problems above, it will be much easier to obtain the result if one knows that

$$(1+x)^{\frac{1}{x}} \approx e\left(1 - \frac{x}{2} + \frac{11x^2}{24} - \frac{5x^3}{12} + \cdots\right) \qquad \forall\, 0 < x \ll 1.$$

Problem 10. In L'Hôpital's 1696 calculus textbook, he illustrated his rule using the limit of the function

$$f(x) = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, $a > 0$. Find this limit.

Problem 11. For what values of a and b is the following equations true?

1. $\lim\limits_{x\to 0} \left(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\right) = 0$.

2. $\lim\limits_{x\to 0} \left(\dfrac{\tan 2x}{x^3} + \dfrac{a}{x^2} + \dfrac{\sin bx}{x}\right) = 0$.