Exercise Problem Sets 1

Sept. 11 2026

Problem 1-5 是第零章與三角函數有關的習題。

Problem 1. Let θ be a real number such that $\sin\dfrac{\theta}{2} \neq 0$.

1. Show that $\displaystyle\sum_{k=1}^{n} \sin(k\theta) = \frac{\cos\frac{1}{2}\theta - \cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$.

2. Show that $\displaystyle\sum_{k=1}^{n} \cos(k\theta) = \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{\theta}{2}}$.

Problem 2. Let θ be a real number such that $\sin\theta \neq 0$. Show that

$$\cos^2\theta + \cos^2(2\theta) + \cdots + \cos^2(n\theta) = \frac{n}{2} + \frac{\sin(n\theta)\cos(n+1)\theta}{2\sin\theta} \qquad \forall\, n \in \mathbb{N}\,.$$

Problem 3. Let θ be a real number and $\theta \neq k\pi$ for all $k \in \mathbb{Z}$（k 非 π 的整數倍）. Show that

$$\cos\theta \cdot \cos(2\theta) \cdot \cdots \cdot \cos(2^n\theta) = \frac{\sin(2^{n+1}\theta)}{2^{n+1}\sin\theta} \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,.$$

Problem 4. Let α, β, γ be real numbers. Suppose that

$$\cos\alpha + \cos\beta + \cos\gamma = 0\,,$$
$$\sin\alpha + \sin\beta + \sin\gamma = 0\,.$$

Show that

1. $\cos(\alpha - \beta) = -\dfrac{1}{2}$.

2. $\cos(\alpha + \beta) = \cos(2\gamma)$.

3. $\sin(\alpha + \beta) = \sin(2\gamma)$.

4. $\cos(2\alpha) + \cos(2\beta) + \cos(2\gamma) = 0$.

5. $\sin(2\alpha) + \sin(2\beta) + \sin(2\gamma) = 0$.

Problem 5. Let θ be a real number such that $t = \tan\dfrac{\theta}{2}$ also be a real number. Show that

$$\sin\theta = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos\theta = \frac{1-t^2}{1+t^2}\,.$$

Note that the two identities above imply that $\tan\theta = \dfrac{2t}{1-t^2}$.

Problem 6. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p},\ p, q \in \mathbb{N} \text{ and } (p, q) = 1\,, \\ 0 & \text{if } x \text{ is irrational}\,. \end{cases}$$

我們已在課堂上約略講解過這題證明的概念，以下是完整證明。

Proof. Let $\varepsilon > 0$ be given. Then there exists a prime number p such that $\dfrac{1}{p} < \varepsilon$. Let $q_1, q_2, \cdots, q_n$ be rational numbers in $\left(\dfrac{c}{2}, \dfrac{3c}{2}\right)$ satisfying

$$q_j = \frac{s}{r}\,, (r, s) = 1, 1 \leqslant r \leqslant p\,,$$

and define $\delta = \dfrac{1}{2}\min\Big(\big\{|c - q_1|, |c - q_2|, \cdots, |c - q_n|\big\}\backslash\{0\}\Big)$. Then $\delta > 0$. Suppose that x satisfies that $0 < |x - c| < \delta$.

1. If $x \in \mathbb{Q}^{\complement}$, then $f(x) = 0$ which shows that $|f(x)| < \varepsilon$.
2. If $x \in \mathbb{Q}$, then $x = \dfrac{s}{r}$ for some natural numbers r, s satisfying $(r, s) = 1$. By the choice of δ, we find that $r > p$; thus
$$\big|f(x)\big| = \frac{1}{r} < \frac{1}{p} < \varepsilon\,.$$

In either case, $\big|f(x)\big| < \varepsilon$; thus we establish that

$$\big|f(x) - 0\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

Therefore, $\lim\limits_{x \to c} f(x) = 0$. □

這題希望同學想的是：

1. 為何只取 $\left(\dfrac{c}{2}, \dfrac{3c}{2}\right)$ 這個區間之內分子不大於 p 的有理數？跟接下來的證明的關係是什麼？
2. δ 的定義中為什麼要扣掉 $\{0\}$？不扣掉零元素證明可以完成嗎？
3. δ 的定義中為什麼要除以 2？不除以 2 證明可以完成嗎？

Problem 7. Let f be given in Problem 6 and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} f(x) & \text{if } x > 0\,, \\ f(-x) & \text{if } x < 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Find $\lim\limits_{x \to 0} g(x)$.

Problem 8. Let f be a function defined on an open interval containing c and $\lim\limits_{x \to c} f(x)$ exists. Show that there exist $\delta > 0$ and $M > 0$ such that

$$\big|f(x)\big| \leqslant M \qquad \text{whenever} \quad |x - c| < \delta\,.$$

Problem 9. Let f, g be a function defined on an open interval containing c (except possibly at c), and $f(x) \leqslant g(x)$ for all $x \neq c$. Prove that if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$ both exist, then $L \leqslant K$.

Problem 10. 1. Suppose that $\lim_{x\to 2} \dfrac{f(x)-5}{x-2} = 3$. Find $\lim_{x\to 2} f(x)$.

2. Suppose that $\lim_{x\to 2} \dfrac{f(x)-5}{x-2} = 4$. Find $\lim_{x\to 2} f(x)$.

3. Suppose that $\lim_{x\to c} \dfrac{f(x)-p(x)}{x-c} = L$ exists, where p is a polynomial function. Find $\lim_{x\to c} f(x)$.

Problem 11. In class you are given the identity $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Compute the following limits:

1. $\lim_{x\to 0} \dfrac{\sin(x^2)}{x}$.
2. $\lim_{x\to 0} \dfrac{x \sin x}{1-\cos x}$.
3. $\lim_{x\to 0} \dfrac{\sin(\sin(\sin x))}{x}$.
4. $\lim_{x\to 0} \dfrac{\sin(x+c)-\sin c}{x}$, where c is a real number.