Exercise Problem Sets 6

Oct. 15. 2026

Problem 1. 1. Let $f, g : (a, b) \to \mathbb{R}$ be functions and $f'(x) = g'(x)$. Show that there exists a constant C such that $f(x) = g(x) + C$.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying that $f'(x) = 3x^2 + 4\cos x$ and $f(0) = 0$. Find $f(x)$.

Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that f has only one critical point $c \in (a, b)$.

1. Show that if $f(c)$ is a local extremum of f, then $f(c)$ is an absolute extremum of f.

2. Show that if $f(c)$ is the absolute minimum of f, then $f(x) > f(c)$ for all $x \in [a, b]$ and $x \neq c$. Similarly, show that if $f(c)$ is the absolute maximum of f, then $f(x) < f(c)$ for all $x \in [a, b]$ and $x \neq c$.

Problem 3. 1. If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\dfrac{2\sqrt{3}}{9}$ at $x = \dfrac{1}{\sqrt{3}}$, what are the values of a and b?

2. Which of the tangent lines to the curve in part (1) has the smallest slope?

Problem 4. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

Problem 5. Suppose f is an odd function (that is, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$) and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = \dfrac{f(b)}{b}$.

Problem 6. Let a be a non-zero real number, $\lambda > 0$, and f be a function defined by

$$f(x) = \frac{1}{2}(x - a)^2 + \lambda|x|\,.$$

1. Find critical points of f. Note that the critical point of f varies with a and λ (so you need to provide critical points according to the values of a and λ).

2. Show that f attains its absolute minimum at the point $x = \operatorname{sgn}(a)\max\{|a| - \lambda, 0\}$, where sgn is the sign function.