Exercise Problem Sets 7

Oct. 23. 2026

Problem 1. Show that $2\sqrt{x} > 3 - \dfrac{1}{x}$ for all $x > 1$.

Problem 2. Show that $\sqrt{b} - \sqrt{a} < \dfrac{b-a}{2\sqrt{a}}$ for all $0 < a < b$.

Problem 3. Show that for all (rational numbers) $p, q \in (1, \infty)$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$ac + bd \leqslant (a^p + b^p)^{\frac{1}{p}} (c^q + d^q)^{\frac{1}{q}} \qquad \forall\, a, b, c, d > 0\,.$$

Hint: Let $x = \dfrac{a}{b}$ and $y = \dfrac{d}{c}$.

Problem 4. Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \qquad \forall\, x \geqslant 0\,,$$

$$1 - \frac{x^2}{2!} + \cdots + \frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \cdots + \frac{x^{4k}}{(4k)!} \qquad \forall\, x \geqslant 0\,.$$

Problem 5. （不要用交叉相乘）Show that for all $k \in \mathbb{N} \cup \{0\}$,

$$1 - x + x^2 - x^3 + \cdots + x^{2k} - x^{2k+1} \leqslant \frac{1}{1+x} \leqslant 1 - x + x^2 - x^3 + \cdots + x^{2k} \qquad \forall\, x \geqslant 0\,.$$

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying that $f'(x) = f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$.

1. （不要試著找出 f 而是直接用 f 的性質）Show that f is increasing on $\mathbb{R}$.
2. Show that if $k \in \mathbb{N} \cup \{0\}$, then $f(x) \geqslant 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for all $x \geqslant 0$.
3. Show that if $k \in \mathbb{N} \cup \{0\}$, then

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!} \leqslant f(x) \leqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2k}}{(2k)!} \qquad \forall\, x \leqslant 0\,.$$

Hint: 1. Show that f^2 is increasing on $\mathbb{R}$ and argue that f is also increasing on $\mathbb{R}$.

Problem 7. 1. The function

$$f(x) = \begin{cases} 0 & \text{if } x = 0\,, \\ 1 - x & \text{if } 0 < x \leqslant 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

2. Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

Problem 8. Find the minimum value of

$$\left|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x\right|$$

for real numbers x.
Hint: Let $t = \sin x + \cos x$.

Problem 9. Let $f, g : (a, b) \to \mathbb{R}$ be twice differentiable functions such that $f''(x) \neq 0$ and $g''(x) \neq 0$ for all $x \in (a, b)$. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).

Problem 10. For what values of a and b is $(2, 2.5)$ an inflection point of the curve $x^2 + ax + by = 0$? What additional inflection points does the curve have?

Problem 11. Let $f : [a, b] \to \mathbb{R}$ be a function, and f is Riemann integrable on $[a, b]$. Show that f must be bounded on $[a, b]$; that is, there exists a real number $M > 0$ such that $|f(x)| \leqslant M$ for all $a \leqslant x \leqslant b$.

Problem 12. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b \cos x \, dx$ by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Write down the Riemann sum using the right end-point rule.

(b) Prove that

$$\sum_{i=1}^{n} \cos(a + id) = \frac{\sin\left[a + \left(n + \frac{1}{2}\right)d\right] - \sin\left(a + \frac{d}{2}\right)}{2 \sin \frac{d}{2}}. \tag{0.1}$$

Hint: Use the sum and difference formula $\sin(\vartheta + \varphi) - \sin(\vartheta - \varphi) = 2 \sin \vartheta \cos \varphi$.

(c) Use (0.1) to simplify the Riemann sum in (a), and find the limit of the Riemann sum as n approaches infinity. Show that

$$\int_a^b \cos x \, dx = \sin b - \sin a \, .$$

Problem 13. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b x^N \, dx$, where N is a non-negative integer, by the following steps.

(a) Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$. Show that the Riemann sum using the right end-point rule is given by

$$I_n = \sum_{k=0}^{N} \left[C_k^N a^{N-k} (b - a)^{k+1} \Big(\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k \Big) \right],$$

where $C_k^N = \dfrac{N!}{k!(N-k)!}$.

(b) Show that

$$\sum_{i=1}^{n} i^{\,k} = \frac{1}{k+1}(n+1)^{k+1} - \frac{1}{k+1}\Big[C_{k-1}^{k+1}\sum_{i=1}^{n} i^{\,k-1} + \cdots + C_{1}^{k+1}\sum_{i=1}^{n} i + (n+1)\Big]. \tag{0.2}$$

Hint: Expand $(j+1)^k$ for $j = 0, 1, 2, \cdots, n$ by the binomial expansion formula, and sum over j to obtain the equality above.

(c) Use (0.2) to show that $\lim\limits_{n\to\infty} \dfrac{1}{n^{k+1}} \sum\limits_{i=1}^{n} i^k = \dfrac{1}{k+1}$ for each $k \in \mathbb{N}$.

(d) Use the limit in (c) to find the limit of the Riemann sum in (a) by passing to the limit as n approaches infinity. Simplify the result to show that

$$\int_a^b x^N \, dx = \frac{b^{N+1} - a^{N+1}}{N+1}.$$

Hint: (c) By induction!

Problem 14. In class we have used the limit of Riemann sums to compute the integral $\int_0^{\pi} x \cos x \, dx$. Find this integral by completing what we did in class.