Exercise Problem Sets 8

Oct. 30. 2026

Problem 1. Find an anti-derivative of the function $y = x\cos x$ using (the limit of) Riemann sums. **Hint**: Use the regular partition to form Riemann sums and use the result in Problem 4 of Exercise 4.

Problem 2. Recall that in Problem 5 of Exercise 4 we have "shown" that there exists a number $e > 1$ such that the logarithmic function $y = \log_e x$ satisfies

$$\frac{d}{dx}\log_e x = \frac{1}{x} \qquad \forall\, x > 0\,.$$

In this exercise you are asked to compute $\displaystyle\int_1^b \log_e x\, dx$ using the Riemann sum by the following steps.

1. (Use the implicit differentiation to) show that $\dfrac{d}{dx}e^x = e^x$. Also show that $\dfrac{d}{dx}a^x = a^x \log_e a$.
2. Partition $[1, b]$ into n sub-intervals by $x_i = r^i$, where $1 \leqslant i \leqslant n$ and $r = b^{\frac{1}{n}}$. Show that the Riemann sum given by the right end-point rule is
$$(r-1)\log_e r \sum_{i=1}^{n} i r^{i-1}\,. \tag{0.1}$$
3. Use (0.1) and the formula in Problem 4 of Exercise 4 to simplify the Riemann sum given above and show that the Riemann sum is
$$\frac{nbr - nb - b + 1}{n(r-1)}\log_e b = \Big[b - \frac{b-1}{n(r-1)}\Big]\log_e b.$$
4. Pass to the limit as $n \to \infty$ to show that
$$\int_1^b \log_e x\, dx = b\log_e b - b + 1\,.$$
5. Verify that $f(x) = x\log_e x - x$ is an anti-derivative of $y = \log_e x$.

Problem 3. Determine the following limits by identifying the limits as limits of certain Riemann sums so that the limits are the same as certain integrals.

1. $\displaystyle\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}}{n^{\frac{3}{2}}}$.
2. $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}}\Big(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\Big)$.
3. $\displaystyle\lim_{n\to\infty} \Big[\frac{1}{\sqrt{n^2+2n}} + \frac{1}{\sqrt{n^2+4n}} + \frac{1}{\sqrt{n^2+6n}} + \cdots + \frac{1}{\sqrt{n^2+2n^2}}\Big]$.

Problem 4. Suppose that $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable on $[a, b]$. Under the fact that fg is Riemann integrable on $[a, b]$, show that if $p, q \in (1, \infty)$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then

$$\int_a^b f(x)g(x)\,dx \leqslant \Big(\int_a^b \big|f(x)\big|^p\,dx \Big)^{\frac{1}{p}} \Big(\int_a^b \big|g(x)\big|^q\,dx \Big)^{\frac{1}{q}}.$$

Hint: First making use of Young's inequality (that we talked about in class) to show the Hölder inequality

$$\Big| \sum_{k=1}^n a_k b_k \Big| \leqslant \Big(\sum_{k=1}^n |a_k|^p \Big)^{\frac{1}{p}} \Big(\sum_{k=1}^n |b_k|^q \Big)^{\frac{1}{q}}.$$

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$, and $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Show that

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a).$$

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be a function satisfying that

$$\big|f(x) - f(y)\big| \leqslant M|x - y| \qquad \forall\, x, y \in [0, 1].$$

Under the fact that f is Riemann integrable on $[0, 1]$, show that

$$\Big| \int_0^1 f(x)\,dx - \frac{1}{n} \sum_{i=1}^n f\big(\frac{i}{n}\big) \Big| < \frac{M}{2n}.$$

Problem 7. Use Problem 7 of Exercise 5 to find the integral $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2 + 1}\,dx$.