

Resource-sharing system scheduling and circular chromatic number

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Abstract

A graph G is used as a model for a resource sharing system, where each vertex represents a process and an edge joining two vertices means that the corresponding processes share a resource. A scheduling of G is a mapping $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$, where $f(i)$ consists of processes that are operating at round i . The rate of f is defined as $\text{rate}(f) = \limsup_{n \rightarrow \infty} \sum_{i=1}^n |f(i)|/n|V(G)|$, which is the average fraction of operating processes at each round. A scheduling is fair if adjacent vertices alternate their turns in operating. The operating rate $\gamma^*(G)$ of G is the maximum rate of a fair scheduling. Fair schedulings of a graph was first studied by Barbosa and Gafni. They introduced the method of “scheduling by edge reversal” which derives a fair scheduling through an acyclic orientation. Through scheduling by edge reversal, Barbosa and Gafni related $\gamma^*(G)$ to the structure of acyclic orientations of G . We point out that this relation implies that $\gamma^*(G)$ is equal to the reciprocal of the circular chromatic number of G . Both circular coloring and scheduling by edge reversal have been studied extensively in the past decade. The former by graph theorists, and the latter by computer scientists. However, it seems that neither side knew the existence of the other. This paper intends to build a connection between

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the two sides. We show that certain open problems concerning scheduling by edge reversal are indeed solved under the language of circular coloring. In the study of fair scheduling, Barbosa and Gafni defined a variation of multiple coloring of graphs: the interleaved p -color, q -tuple colorings. We formulate the interleaved coloring as a graph homomorphism problem. In the study of circular chromatic number, Bondy and Hell defined (p, q) -colorings and also formulated it as a graph homomorphism problem. We prove that the target graph for the interleaved p -color, q -tuple coloring and the target graph of (p, q) -coloring are homomorphically equivalent. This gives another proof of the fact that $\gamma^*(G) = 1/\chi_c(G)$. Moreover, the proof gives an explicit formula which deduces an optimal circular coloring of G from an optimal interleaved coloring of G , and vice versa. This paper also introduces two other schedulings of a graph, the weakly fair scheduling and the strongly fair scheduling. It is proved that the rate of an optimal strongly fair scheduling of a graph G is also equal to the reciprocal of the circular chromatic number of G , and the rate of an optimal weakly fair scheduling of G is equal to the reciprocal of the fractional chromatic number of G . Barbosa and Gafni presented an algorithm that determines the rate $\gamma_o(w)$ of the scheduling induced by an acyclic orientation w of G . By using Karp's minimum mean cycle algorithm, we give a faster algorithm to calculate $\gamma_o(w)$.

1 Introduction

A *resource-sharing system* consists of a set of processes and a set of resources. Each process has access to a subset of resources. A *deadlock* occurs when two processes sharing a resource operate at the same time. A *scheduling* of a resource sharing system is to determine the operating time of each process so that there is no deadlock. Such a system can be modeled as a undirected graph G in which each vertex represents a process and two vertices are adjacent if the corresponding processes share a resource. In this mathematical model a scheduling of a resource-sharing system (a graph) G is a mapping $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$, where $f(i)$ is an independent set of G which consists of the vertices (i.e., processes) that operate at round i . The *rate* of f is defined as $\text{rate}(f) = \limsup_{n \rightarrow \infty} \sum_{i=1}^n |f(i)|/n|V(G)|$, which is the average fraction of operating processes at each round of the scheduling f . So $\text{rate}(f)$ measures the efficiency of a scheduling f .

Different practical problems impose different constraints on a legal scheduling. For $v \in V(G)$, $i \geq 1$ and I an interval of real line, let $f(v, I) = |\{j : j \in I \text{ is a positive integer and } v \in f(j)\}|$. For simplicity, $f(v, i)$ stands for $f(v, (0, i])$, i.e., $f(v, i)$ is the number of rounds till round i in which v operated. We say f is *fair* if for

any two adjacent vertices u, v and for any interval $I = (i, j]$, $|f(u, I) - f(v, I)| \leq 1$. We say f is *strongly fair* if for any two (not necessarily adjacent) vertices u, v and for any interval $I = (i, j]$, $|f(u, I) - f(v, I)| \leq 1$. We say f is *weakly fair* if for any two vertices u, v and for every integer i , $|f(u, i) - f(v, i)|$ is bounded by a constant. Note that if f is a weakly fair scheduling, then in the long run, each vertex operates almost the same number of times, and hence $\limsup_{n \rightarrow \infty} f(v, n)/n = \text{rate}(f)$ for each vertex v . In particular, if the weakly fair scheduling f is periodic with period p , i.e., $f(p+i) = f(i)$ for all i , then within a period, each vertex operates $p \times \text{rate}(f)$ times. A scheduling f is fair if and only if adjacent vertices alternate their turns in operation. A scheduling f is strongly fair if and only if any two vertices almost alternate their turns in operation, i.e., if x operates at round i but y does not, then after round i , y must operate before or at the same time as x operates again.

Example: Let C_7 be the cycle with vertex set $\{v_0, v_1, \dots, v_6\}$. Let f, g, h be defined as follows: $f(i) = \{v_i, v_{i+2}\}$ for $i \geq 0$, $g(i) = \{v_i, v_{i+3}\}$ for $i \geq 0$, and $h(2i) = \{v_i, v_{i+3}\}$, $h(2i+1) = \{v_i, v_{i+4}\}$ for $i \geq 0$. Here the summations are modulo 7. Then h is a weakly fair scheduling but not a fair scheduling, f is a fair scheduling but not a strongly fair scheduling, g is a strongly fair scheduling.

In the remaining, we shall refer a weakly fair scheduling as a WF-scheduling, a fair scheduling as an F-scheduling and a strongly fair scheduling as an SF-scheduling. Given a graph G , let

$$\gamma_w^*(G) = \sup\{\text{rate}(f) : f \text{ is a WF-scheduling of } G\},$$

$$\gamma^*(G) = \sup\{\text{rate}(f) : f \text{ is an F-scheduling of } G\},$$

$$\gamma_s^*(G) = \sup\{\text{rate}(f) : f \text{ is an SF-scheduling of } G\}.$$

A WF-scheduling (resp. an F-scheduling or an SF-scheduling) f is *optimal* if $\text{rate}(f) = \gamma_w^*(G)$ (resp. $\text{rate}(f) = \gamma^*(G)$, or $\text{rate}(f) = \gamma_s^*(G)$). It follows from the definition that an SF-scheduling is an F-scheduling, and if G is finite and connected (which we shall assume in this paper) then an F-scheduling is a WF-scheduling. Therefore

$$\gamma_w^*(G) \geq \gamma^*(G) \geq \gamma_s^*(G).$$

This paper relates the problem of finding optimal F-schedulings, optimal WF-schedulings and optimal SF-schedulings of a graph G to the circular coloring and the

fractional coloring of G . The circular coloring and the fractional coloring of graphs are variations of the vertex coloring of graphs.

Given a real number $r \geq 2$, a *circular r -coloring* of a graph G is a mapping $f : V(G) \rightarrow [0, r)$ such that for every edge (x, y) of G , $1 \leq |f(x) - f(y)| \leq r - 1$. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of those r for which G has a circular r -coloring. The *chromatic number* $\chi(G)$ of a graph G is the least integer k such that there is mapping (coloring) $f : V(G) \rightarrow \{0, 1, \dots, k - 1\}$ such that for every edge (x, y) of G , $f(x) \neq f(y)$. It is obvious that such a mapping f is a circular k -coloring of G . Therefore $\chi_c(G) \leq \chi(G)$ for any graph G . Indeed, it is known [31] that for any graph G , $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. So $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$. A circular r -coloring is also called an r -coloring.

Let $\mathcal{I}_G$ denote the collection of independent sets of a graph G . A *fractional coloring* of G is a mapping $f : \mathcal{I}_G \rightarrow [0, 1]$ such that for each vertex $v \in V(G)$, $\sum_{X \in \mathcal{I}_G, v \in X} f(X) = 1$. The *weight* $w(f)$ of a fractional coloring f is $w(f) = \sum_{X \in \mathcal{I}_G} f(X)$. The *fractional chromatic number* $\chi_f(G)$ of G is the least weight of a fractional coloring of G . If the range of a fractional coloring f is restricted to $\{0, 1\}$, i.e., for each $X \in \mathcal{I}_G$, $f(X) = 0$ or 1 , then f is indeed a coloring of G with $w(f)$ as the number of used colors. So $\chi_f(G) \leq \chi(G)$. It is known [22, 23] that $\chi(G)/\chi_f(G)$ could be arbitrarily large.

The fractional chromatic number of a graph can also be defined in terms of graph homomorphisms. Let G and H be two graphs. A *homomorphism* from G to H is a mapping φ from $V(G)$ to $V(H)$ such that $(\varphi(x), \varphi(y)) \in E(H)$ whenever $(x, y) \in E(G)$. If there exists a homomorphism from G to H , then we say that G is *homomorphic* to H , and write $G \rightarrow H$. We say two graphs G and H are *homomorphically equivalent* if $G \rightarrow H$ and $H \rightarrow G$. Given integers p, q such that $p \geq 2q$. Let $S_p = \{1, 2, \dots, p\}$ and denote by $\binom{S_p}{q}$ the family of q -subsets of S_p . The *Kneser graph* $K(p, q)$ has vertex set $\binom{S_p}{q}$, and in which two vertices A and B are adjacent if and only if $A \cap B = \emptyset$. The fractional chromatic number $\chi_f(G)$ of G can be defined as $\chi_f(G) = \inf\{p/q : G \rightarrow K(p, q)\}$ [25].

By combining a result obtained by Barbosa and Gafni [6] concerning the relation between the operating rate $\gamma^*(G)$ and acyclic orientations of G , and a result obtained by Goddyn, Tarsi and Zhang [15] concerning the relation between the circular chro-

matic number $\chi_c(G)$ and acyclic orientations of G , we conclude that $\chi_c(G) = 1/\gamma^*(G)$. So $\chi_c(G)$ and $\gamma^*(G)$ concern the same graph property. It is interesting to note that the parameters $\chi_c(G)$ and $\gamma^*(G)$ were introduced at about the same time. The star chromatic number (an equivalent of the circular chromatic number) was introduced by Vince [29] in 1988. The parameter $\gamma^*(G)$ was introduced by Barbosa [2] in 1986, in his Ph.D. thesis, and as far as we know, the first publication of a detailed discussion of $\gamma^*(G)$ was in 1989 [6] (a condensed version [5] of this paper appeared in 1987). Circular coloring of graphs has attracted considerable attention in the last decade. The parameter $\gamma^*(G)$ did not attract much attention, however, the related concept of scheduling by edge reversal has been studied extensively by computer scientists.

Although circular coloring and scheduling by edge reversal are closely related, each of the two topics have been developing independently. This paper intends to build a bridge between the two sides. In Section 2, we survey some results on circular coloring and scheduling by edge reversal. Some of the results concerning circular coloring provide answers to questions asked by Barbosa and Gafni in [6]. As a byproduct of the discussion of the relation between circular coloring and fair scheduling, we prove that $\chi_c(G) = 1/\gamma_s^*(G)$, which implies that an optimal strongly fair scheduling of G has the same rate as an optimal fair scheduling of G .

In the study of fair scheduling, Barbosa and Gafni defined a variation of multiple coloring of graphs: the interleaved p -color q -tuple colorings. In Section 3, we formulate the interleaved coloring as a graph homomorphism problem. In the study of circular chromatic number, Bondy and Hell defined (p, q) -colorings and also formulated it as a graph homomorphism problem. We prove that the target graph for the interleaved p -color, q -tuple coloring and the target graph of (p, q) -coloring are homomorphically equivalent. This gives another proof of the fact that $\gamma^*(G) = 1/\chi_c(G)$. Moreover, the proof gives an explicit formula which deduces an optimal circular coloring of G from an optimal interleaved coloring of G , and vice versa.

Given an acyclic orientation w of a graph G , denote by $\gamma_o(w)$ the rate of the scheduling by edge reversal generated by w . Barbosa and Gafni [6] designed an algorithm that determines $\gamma_o(w)$ in time $O(n^6)$, where n is the number of vertices. In Section 4, based on the relation between $\gamma_o(w)$ and the *minimum mean cycle problem* [21], we present an algorithm which determines $\gamma_o(G)$ in time $O(nm)$, where m is the

number of edges.

In Section 5, we consider weakly fair scheduling and prove that $\chi_f(G) = 1/\gamma_w^*(G)$. It is known [31] that $\chi_c(G) \geq \chi_f(G)$ for any graph G and the ratio $\chi_c(G)/\chi_f(G)$ could be arbitrarily large. So the rate of an optimal weakly fair scheduling could be much higher than the rate of a fair scheduling.

2 Fair scheduling and circular coloring

To find a fair scheduling of maximum rate, Barbosa and Gafni [6] introduced a scheduling mechanism called *scheduling by edge reversal* which functions as follows: For an acyclic orientation w of a graph G , denote by $\text{sink}(w)$ the set of sinks of w . Starting from an acyclic orientation w of G , we obtain a sequence of acyclic orientations of G $w_1, w_2, w_3, \dots$ in such a way that $w_1 = w$, and for $i \geq 2$, w_i is obtained from w_{i-1} by reversing the orientations of the edges incident to the sinks of w_{i-1} . This sequence is completely determined by w , and hence we say that this sequence is *generated by* w . Now define a mapping $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$ such that $f(i) = \text{sink}(w_i)$. In other words, at round i , the sinks of w_i operate. It is obvious that for each of the orientations w_i , $\text{sink}(w_i)$ is an independent set of G . Moreover, if x and y are adjacent, and x is a sink of w_i and w_{i+t} , then there must be an index j such that $i < j < i + t$ and y is a sink of w_j . So f is a fair scheduling.

The rate of the fair scheduling generated by the orientation w is denoted by $\gamma_o(w)$, i.e.

$$\gamma_o(w) = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n |\text{sink}(w_i)|}{n|V(G)|}.$$

The following result was implicit in [6].

Theorem 1 *For any graph G , $\gamma^*(G) = \max\{\gamma_o(w) : w \text{ is an acyclic orientation of } G\}$.*

Thus finding an optimal fair scheduling of G amounts to find an optimal acyclic orientation w of G in terms of $\gamma_o(w)$. Suppose that w is an orientation of G and C is a closed walk of G . Denote by C_w^+ and C_w^- the set of *forward arcs* and the set of *backward arcs* of C in orientation w , respectively. That is, C_w^+ is the collection of edges of C whose direction in w agree with the direction of the traversal of the closed walk. The following result was proved in [6].

Theorem 2 *For an acyclic orientation w of G ,*

$$\gamma_o(w) = \min_C \frac{|C_w^+|}{|C|},$$

where the minimum is taken over all closed walks C of G .

It is natural that scheduling problems are related to graph colorings. Barbosa and Gafni [6] proved that for any graph G , $1/\chi(G) \leq \gamma^*(G) \leq 1/\chi_f(G)$, where $\chi(G)$ and $\chi_f(G)$ are the chromatic number and the fractional chromatic number of G , respectively. It turns out that the graph coloring which is more closely related to the parameter $\gamma^*(G)$ is the circular coloring defined in Section 1. The following result was proved by Goddyn, Tarsi and Zhang [15].

Theorem 3 *Suppose G is a graph. For an acyclic orientation w of G , let $\xi(w) = \max\{|C|/|C_w^+| : C \text{ is a closed walk of } G\}$. Then*

$$\chi_c(G) = \min\{\xi(w) : w \text{ is an acyclic orientation of } G\}.$$

As a consequence of Theorems 1, 2 and 3, we have the following result:

Theorem 4 *For any graph G , $\chi_c(G) = 1/\gamma^*(G)$.*

The circular chromatic number $\chi_c(G)$ is a very natural refinement of the chromatic number $\chi(G)$. Many problems concerning chromatic number of a graph have been studied in the framework of the circular chromatic number [31]. On the other hand, $\gamma^*(G)$ arose from a very natural scheduling problem. The equality $\chi_c(G) = 1/\gamma^*(G)$ shows that the two parameters concern the same graph property.

The equivalence of the two parameters provides answers to some questions asked by Barbosa and Gafni in [6]. It was proved in [6] that it is NP-complete to determine if $\gamma^*(G) \geq 1/n$ for any integer $n \geq 5$. They asked if there are some rationals r such that it can be determined in polynomial time whether $\gamma^*(G) \geq r$. Since it is known that for any real number $s > 2$, the problem to determine whether $\chi_c(G) \leq s$ is NP-complete [7, 31], so it is NP-complete to determine if $\gamma^*(G) \geq r$ for any $r < 1/2$. Another question asked by Barbosa and Gafni [6] is whether an approximation for $\gamma^*(G)$ would translate into an approximation for $\chi(G)$ and $\chi_f(G)$. The difference between $1/\gamma^*(G)$ and $\chi_f(G)$ could be arbitrarily large [22, 23]. So $\gamma^*(G)$ does not give a good

approximation of $\chi_f(G)$. On the hand, as $\chi(G) = \lceil \chi_c(G) \rceil$, any approximation for $\gamma^*(G)$ automatically induces an approximation for $\chi(G)$. However the vertex coloring problem is hard to approximate in general [9].

Although the equality $\chi_c(G) = 1/\gamma^*(G)$ follows from Theorems 1, 2 and 3, Theorem 1 was not explicitly stated and proved, and the proof of Theorem 2 in [6] (Theorem 16) is a little bit complicated. In the following, we give a self-contained proof of Theorem 4, which is short and simple and relates scheduling and coloring to tensions of graphs.

First we give another definition of the circular chromatic number of a graph G . Given an orientation w , a mapping $\phi : E(G) \rightarrow \mathcal{R}$ is called a *tension* of (G, w) if for each cycle C ,

$$\sum_{e \in C_w^+} \phi(e) = \sum_{e \in C_w^-} \phi(e).$$

For a real number $r \geq 2$, a tension ϕ is called an *r-tension* if for each edge e in G , $1 \leq |\phi(e)| \leq r - 1$. If an orientation w of G admits an *r-tension* ϕ , then every orientation of G admits an *r-tension*. Indeed, if w' is an orientation obtained from w by reversing the orientation of an edge e , and ϕ is an *r-tension* with respect to w , then by changing $\phi(e)$ to $-\phi(e)$, we obtain an *r-tension* with respect to the new orientation w' . In particular, by changing the orientations of all the edges e for which $\phi(e) < 0$, we obtain an orientation w^* which admits a *positive r-tension* ϕ^* , i.e. $\phi^*(e) > 0$ for every edge e .

The relation between *r-tension* and circular chromatic number is shown in the following well-known lemma. A short proof is included for the completeness of the paper.

Lemma 5 *A graph G has an r -coloring if and only if for some orientation (and hence for every orientation) w of G , (G, w) admits an r -tension.*

Proof. Let $f : V(G) \rightarrow [0, r)$ be an *r-coloring* of G . Let w be an arbitrary orientation of G . For each edge $e = (x, y)$ of G , let $\phi_f(e) = f(y) - f(x)$ if e is oriented from x to y . Then it is easy to verify that $\phi_f : E(G) \rightarrow \mathcal{R}$ is an *r-tension*. Conversely, assume that $\phi : E(G) \rightarrow \mathcal{R}$ is an *r-tension* of (G, w) . Let x_0 be an arbitrary fixed vertex of G . For each vertex x of G , let $W(x)$ be a walk from x_0 to x . We define a mapping $f : V(G) \rightarrow \mathcal{R}$ by letting $f(x) = \sum_{e \in W(x)_w^+} \phi(e) - \sum_{e \in W(x)_w^-} \phi(e)$. Note

that the mapping f is well-defined, since ϕ is a tension and hence the value $f(x)$ does not depend on the choice of the walk $W(x)$. It is easy to verify that $c(x) = f(x) \pmod{r}$ defines an r -coloring of G . $\blacksquare$

An acyclic orientation w on a graph G is called a (k, d) -orientation if for every closed walk C in G , $\min\{|C_w^+|/|C|, |C_w^-|/|C|\} \geq d/k$.

Lemma 6 *Let w be an acyclic orientation of G . The following are equivalent.*

1. w admits a positive k/d -tension.
2. w is a (k, d) -orientation.
3. $\gamma_o(w) \geq d/k$.

Proof. (1) $\Rightarrow$ (2): Assume ϕ is a positive k/d -tension of G . Then for any cycle C in G ,

$$|C_w^+| \leq \sum_{e \in C_w^+} \phi(e) = \sum_{e \in C_w^-} \phi(e) \leq (k/d - 1)|C_w^-|.$$

Therefore $|C|/|C_w^-| = 1 + |C_w^+|/|C_w^-| \leq k/d$ and hence w is a (k, d) -orientation of G .

(2) $\Rightarrow$ (3): Let $\{w_i\}_{i=1}^\infty$ be the sequence of acyclic orientations generated by w . Since there are finite many orientations of G , there is an integer p and an index i_0 such that $w_{i_0+p} = w_{i_0}$. The smallest such an integer p is called the *period* of $\{w_i\}_{i=1}^\infty$. Observe that if w'' is an orientation of G obtained from another orientation w' by reversing the orientations of all the edges incident to a vertex v , then for any closed walk C of G , $|C_{w''}^+| = |C_{w'}^+|$ and $|C_{w''}^-| = |C_{w'}^-|$. So $|C_{w_i}^+| = |C_{w_j}^+|$ and $|C_{w_i}^-| = |C_{w_j}^-|$ for all i, j . Therefore without loss of generality, we may assume $i_0 = 1$ (if $i_0 > 1$, we prove (3) for the acyclic orientation w_{i_0} instead w). Then we have $w_{p+i} = w_i$ for all $i \geq 1$. Note that each vertex of G appears as a sink the same number of times in a period $w_1, w_2, \dots, w_p$. Assume that each vertex appears as a sink m times in a period. Then $\gamma_o(w) = m/p$. We shall prove that there is a closed walk C with $|C_w^+|/|C| = m/p$, which implies that $\gamma_o(w) = m/p \geq d/k$.

For any $i \geq 2$, each sink of w_i is adjacent to a sink of w_{i-1} . Let $n = |V(G)|$, let v_{pn+1} be an arbitrary sink of w_{pn+1} . Suppose $v_i \in \text{sink}(w_i)$ is chosen, let v_{i-1} be a sink of w_{i-1} which is adjacent to v_i . Thus we obtain a sequence

$$v_{pn+1}v_{pn} \cdots v_{p+1}v_pv_{p-1} \cdots v_1$$

such that $(v_i, v_{i-1}) \in E(G)$ and $v_i \in \text{sink}(w_i)$. Since G has only n vertices, there are integers $0 \leq t' < t \leq n$ such that $v_{pt+1} = v_{pt'+1}$. Without loss of generality, we assume $t' = 0$. Then $W = (v_1, v_2, \dots, v_{p-1}, v_p, v_{p+1}, \dots, v_{pt-1}, v_{pt+1} = v_1)$ is a closed walk of G .

For each vertex x of G and for $1 \leq i \leq pt+1$, let $h(x, i) = \{j : 1 \leq j \leq i, x \in \text{sink}(w_j)\}$, i.e., $h(x, i)$ is the number of times x becomes a sink in the process w_1 is changed to w_i . If $\overrightarrow{xy}$ is an directed edge of $w = w_1$, then y becomes a sink first, and after that x and y alternate their turns as a sink. So at the time y becomes a sink for the j th time, x has been a sink for $(j-1)$ times. At the time x becomes a sink for the j th times, y has been a sink for j times already. Therefore if $\overrightarrow{v_i v_{i+1}}$ is an arc of w , then $h(v_{i+1}, i+1) = h(v_i, i) + 1$. If $\overrightarrow{v_{i+1} v_i}$ is an arc of w , then $h(v_{i+1}, i+1) = h(v_i, i)$. Since $h(v_1, 1) = 1$ and $h(v_{pt+1}, pt+1) = h(v_1, pt+1) = mt+1$, we conclude that $\sum_{j=1}^{pt} (h(v_{j+1}, j+1) - h(v_j, j)) = h(v_{pt+1}, pt+1) - h(v_1, 1) = mt$. This implies that with respect to the orientation w , W contains mt forward edges. So $|W_w^+|/|W| = mt/pt = m/p$.

(3) $\Rightarrow$ (1): Similarly as in the proof of (2) $\Rightarrow$ (3), without loss of generality we may assume that the sequence of orientations $w_1, w_2, \dots$, generated by w is periodic with period p , i.e. $w_i = w_{i+p}$ for $i \geq 1$, and in a period each vertex of G appears as a sink m times. Then $m/p = \gamma_o(w)$. Now fix a vertex x_0 and define

$$f(x) = \sup\{d|W| - k|W_w^+| : W \text{ is a walk from } x_0 \text{ to } x\}$$

for each vertex x . For convenience, we call this value $d|W| - k|W_w^+|$ the *weight* of W in w . We claim that $f(x)$ exists (i.e. it is of finite value) for each vertex x in G . It suffices to prove that the weights of walks from x_0 to x are bounded from above. To show this we claim that if we remove a closed walk C from any (x_0, x) -walk, we obtain an (x_0, x) -walk of no less weight. Indeed, since by hypothesis, $p/m \leq k/d$ then

$$\begin{aligned} d|C| - k|C_w^+| &\leq \frac{d}{m}\{m|C| - p|C_w^+|\} \\ &= \frac{d}{m}\{m|C| - \sum_{i=1}^p |C_{w_i}^+|\} \\ &\leq \frac{d}{m}\{m|C| - \sum_{i=1}^p |\text{sink}(G, w_i) \cap C|\} \\ &= \frac{d}{m}\{m|C| - m|C|\} = 0. \end{aligned}$$

Which completes the proof of claim. For each arc $e = \overrightarrow{xy}$ of w , let $\phi_f(e) = \frac{f(x)}{d} - \frac{f(y)}{d}$. We have $f(y) \geq f(x) + d - k$ and $f(x) \geq f(y) + d$, and hence $1 \leq \phi_f(e) \leq \frac{k}{d} - 1$. It follows from the definition that for any closed walk W , $\sum_{e \in W_w^+} \phi_f(e) = \sum_{W_w^-} \phi_f(e)$. So ϕ_f is a positive k/d -tension of (G, w) . $\blacksquare$

Theorem 4 follows from Lemmas 5 and 6.

3 F-scheduling and circular coloring viewed as graph homomorphisms

In this section we formulate F-scheduling and circular coloring as graph homomorphisms. First we consider F-schedulings. Given integers p, q such that $p \geq 2q$. Recall that $S_p = \{1, 2, \dots, p\}$ and $\binom{S_p}{q}$ is the family of q -subsets of S_p . For two q -subsets $A = \{a_1 < a_2 < \dots < a_q\}$ and $B = \{b_1 < b_2 < \dots < b_q\}$, we say A and B are *interleaved* if either $a_1 < b_1 < a_2 < b_2 < \dots < a_q < b_q$ or $b_1 < a_1 < b_2 < a_2 < \dots < b_q < a_q$. Let $B(p, q)$ be the graph with vertex set $\binom{S_p}{q}$ in which two vertices are adjacent if the corresponding q -subsets are interleaved. Observe that $B(p, q)$ is a subgraph of $K(p, q)$, since interleaved subsets are disjoint.

Lemma 7 *For any graph G , $\gamma^*(G) = \sup\{q/p : G \rightarrow B(p, q)\}$.*

Proof. Suppose that ϕ is a homomorphism from G to $B(p, q)$. Define a mapping $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$ such that $f(i) = \{v \in V(G) : \phi(v) \text{ contains } i\}$ for $1 \leq i \leq p$ and $f(i) = f(i - p)$ for $i \geq p + 1$. It is easy to verify that f is an F-scheduling with $\text{rate}(f) = q/p$. Therefore $\gamma^*(G) \geq \sup\{q/p : G \rightarrow B(p, q)\}$.

To prove the converse, let f be an F-scheduling of G . Define m_n to be the value $\min\{f(v, n) : v \in V(G)\}$ for each integer $n \geq 1$. Define $S_v^n = \{i : f(i) \text{ contains } v \text{ and } f(v, i) \leq m_n\}$ for each $v \in V(G)$. Note that S_v^n is an m_n -subset of $\{1, 2, \dots, n\}$ for each $v \in V(G)$. Since the scheduling f is fair, we have that if (u, v) is an edge of G then the two m_n -subsets S_v^n and S_u^n are interleaved. It follows that there exists a homomorphism from G to $B(n, m_n)$ for any sufficiently large n . Therefore

$$\sup\left\{\frac{q}{p} : G \rightarrow B(p, q)\right\} \geq \limsup_{n \rightarrow \infty} \frac{m_n}{n} \geq \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^n |f(i)| - |V(G)|^2}{n|V(G)|} = \text{rate}(f).$$

Since f is arbitrary, we conclude that $\sup\{q/p : G \rightarrow B(p, q)\} \geq \gamma^*(G)$. $\blacksquare$

Given a graph H , the H -coloring problem is the decision problem with instance a graph G and question whether or not G admits a homomorphism to H . It was proved by Hell and Nešetřil [18] that the H -coloring problem is polynomial if H is bipartite and it is NP -complete if H is nonbipartite. The graph $B(p, q)$ is bipartite if and only if $p = 2q$. So it is NP -complete to determine whether or not $\gamma^*(G) \geq r$ for any $r < 1/2$.

A homomorphism ϕ from a graph G to $B(p, q)$ was called an *interleaved q -tuple p -coloring* of G in [3]. The infimum of the ratio p/q for which G has an interleaved q -tuple p -coloring was called the *interleaved multichromatic number* of G , and was denoted by $\chi_{int}^*(G)$ in [3].

The circular chromatic number of a graph can also be determined by considering graph homomorphisms [7]. For positive integers $p \geq 2q$, let $K_{p/q}$ be the graph with vertex set $[p] = \{0, 1, \dots, p-1\}$ in which (i, j) is an edge if and only if $q \leq |i - j| \leq p - q$. If r is a rational number, then we write $r = p/q$ in such a way that p, q are coprime integers and let $K_r = K_{p/q}$. Note that in defining $K_{p/q}$, we do not require $\gcd(p, q) = 1$. The following lemma is well-known [31] and we include the short proof for the completeness of this paper.

Lemma 8 *For any graph G , $\chi_c(G) = \inf\{p/q : G \rightarrow K_{p/q}\}$.*

Proof. If ϕ is a p/q -coloring of G , then $f(v) = \lfloor \phi(v)q \rfloor$ is a homomorphism from G to $K_{p/q}$. If f is a homomorphism from G to $K_{p/q}$, then $\phi(v) = f(v)/q$ is a p/q -coloring of G . ■

Suppose there is a periodic F-scheduling f of a graph G of rate q/p . It follows from Theorem 4 that there is a p/q -coloring of G . However, the proofs of Theorem 4, Lemmas 7 and 8 do not give explicitly a p/q -coloring of G . To obtain an explicit formula for a p/q -coloring of G , we need an explicit homomorphism from $B(p, q)$ to $K_{p/q}$.

In the remainder of this section, we present an explicit homomorphism from $B(p, q)$ to $K_{p/q}$ and a homomorphism from $K_{p/q}$ to $B(p, q)$. This gives another proof of the fact that $\chi_c(G) = 1/\gamma^*(G)$. Moreover, it gives an explicit formula which derives a p/q -coloring of G from a periodic fair scheduling f of rate p/q , and conversely, an explicit formula that derives from a p/q -coloring of G a periodic fair scheduling f of rate p/q .

Lemma 9 For any $p \geq 2q \geq 2$, $K_{p/q}$ and $B(p, q)$ are homomorphically equivalent.

Proof. Let $f : [p] \rightarrow \binom{[p]}{q}$ be defined as $f(i) = \{\lfloor \frac{i+tp}{q} \rfloor : t = 0, 1, \dots, q-1\}$. Then f is a homomorphism from $K_{p/q}$ to $B(p, q)$. Indeed, if (i, j) is an edge of $K_{p/q}$, then $q \leq |i - j| \leq p - q$. Without loss of generality, assume that $i > j$. Then for $t = 0, 1, \dots, q-1$,

$$\lfloor (j+tp)/q \rfloor < \lfloor (j+q+tp)/q \rfloor \leq \lfloor (i+tp)/q \rfloor < \lfloor (i+q+tp)/q \rfloor \leq \lfloor (j+(t+1)p)/q \rfloor.$$

So $f(i)$ and $f(j)$ are interleaved, and hence $(f(i), f(j))$ is an edge of $B(p, q)$.

Suppose that $A \in \binom{[p]}{q}$ and $A = \{a_1 < a_2 < \dots < a_q\}$. Let i_A be the index such that $a_{i_A} - i_A p/q \geq a_j - j p/q$ for all j . Let $g : \binom{[p]}{q} \rightarrow [p]$ be defined as $g(A) = a_{i_A} q \pmod{p}$. We claim that g is a homomorphism from $B(p, q)$ to $K_{p/q}$.

Assume that $A = \{a_1 < a_2 < \dots < a_q\}$ and $B = \{b_1 < b_2 < \dots < b_q\}$ are adjacent in $B(p, q)$. Without loss of generality, assume that $a_1 < b_1 < a_2 < b_2 < \dots < a_q < b_q$.

Case 1: $i_A \leq i_B$. If $i_B \leq q-1$, then

$$b_{i_B} - b_{i_A} + 1 \leq b_{i_B} - a_{i_A} \leq a_{i_B+1} - a_{i_A} - 1.$$

As $b_{i_B} - i_B p/q \geq b_{i_A} - i_A p/q$ and $a_{i_A} - i_A p/q \geq a_{i_B+1} - (i_B+1)p/q$, we conclude that $(b_{i_B} - b_{i_A})q \geq (i_B - i_A)p$ and $(a_{i_B+1} - a_{i_A})q \leq (i_B + 1 - i_A)p$. Thus

$$\begin{aligned} (i_B - i_A)p + q &\leq (b_{i_B} - b_{i_A} + 1)q \\ &\leq (b_{i_B} - a_{i_A})q \\ &\leq (a_{i_B+1} - a_{i_A} - 1)q \\ &\leq (i_B - i_A)p + p - q. \end{aligned}$$

Therefore

$$q \leq (b_{i_B} - a_{i_A})q \pmod{p} \leq p - q,$$

or equivalently $q \leq (a_{i_A} - b_{i_B})q \pmod{p} \leq p - q$.

Hence $g(B)$ is adjacent to $g(A)$ in $K_{p/q}$. If $i_B = q$, then replace a_{i_B+1} by p in the calculation above, we obtain the same conclusion.

Case 2: $i_A > i_B$. Then

$$a_{i_A} - a_{i_B+1} + 1 \leq a_{i_A} - b_{i_B} \leq b_{i_A} - b_{i_B} - 1.$$

As $b_{i_B} - i_B p/q \geq b_{i_A} - i_A p/q$ and $a_{i_A} - i_A p/q \geq a_{i_B+1} - (i_B + 1)p/q$, we conclude that $(b_{i_A} - b_{i_B})q \leq (i_A - i_B)p$, and $(a_{i_A} - a_{i_B+1})q \geq (i_A - i_B - 1)p$. Therefore

$$\begin{aligned}
(i_A - i_B - 1)p + q &\leq (a_{i_A} - a_{i_B+1} + 1)q \\
&\leq (a_{i_A} - b_{i_B})q \\
&\leq (b_{i_A} - b_{i_B} - 1)q \\
&\leq (i_A - i_B - 1)p + p - q.
\end{aligned}$$

This again implies that

$$q \leq (a_{i_A} - b_{i_B})q \pmod{p} \leq p - q.$$

Hence $g(B)$ is adjacent to $g(A)$ in $K_{p/q}$. ■

A graph H is called a *core graph* if H is not homomorphic to any of its proper subgraphs. If H is a subgraph of G such that $G \rightarrow H$ and H is a core graph, then we say that H is a *core* of G . It is well-known that every finite graph has a unique core (up to isomorphism). The proof of Theorem 9 shows that $K_{p/q}$ is a subgraph of $B(p, q)$. If $(p, q) = 1$, then it is known [31] that $K_{p/q}$ is a core graph. Hence $K_{p/q}$ is the core of $B(p, q)$.

Theorem 10 *For any graph G ,*

$$\gamma^*(G) = \gamma_s^*(G) = \frac{1}{\chi_c(G)}.$$

Proof. We prove this theorem by showing that

$$\gamma^*(G) \geq \gamma_s^*(G) \geq \frac{1}{\chi_c(G)} = \gamma^*(G).$$

The left-most inequality follows from the definitions of F-scheduling and SF-scheduling. The right-most equality is proved in Theorem 4. It remains to show that $\gamma_s^*(G) \geq 1/\chi_c(G)$. Suppose that G has a circular r -coloring ϕ . For every vertex v in G let $S_v = \{\phi(v) + tr : t = 0, 1, 2, 3, \dots\}$. Define a mapping $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$ such that $f(i) = \{v \in V(G) : S_v \cap [i - 1, i) \neq \emptyset\}$. It is easy to verify that f is an SF-scheduling of G with $\text{rate}(f) = 1/r$. Therefore $\gamma_s^*(G) \geq 1/\chi_c(G)$, which completes the proof. ■

4 An algorithm for determining $\gamma_o(w)$ and an SF-scheduling with rate $\gamma_o(w)$

Barbosa and Gafni [6] designed an algorithm that determines the value $\gamma_o(w)$ for an acyclic orientation w of G in time $O(n^6)$, where n is the number of vertices of G . By adopting Karp's algorithm for the minimum mean cycle problem, we obtain an algorithm that determines $\gamma_o(w)$ in time $O(nm)$, where m is the number of edges of G .

Suppose that D is a digraph with a weight assignment g which assigns to each directed edge e a real number $g(e)$. Let C be a *directed cycle* of D , i.e., $C = e_0 e_1 \cdots e_{\ell-1}$, where $e_i = \overrightarrow{x_i x_{i+1}}$ is a directed edge and $x_\ell = x_0$ and other x_j 's are distinct. The *weight* $g(C)$ of C is defined as $g(C) = \sum_{i=0}^{\ell-1} g(e_i)$. The *mean weight* $m(C)$ of C is defined as $m(C) = g(C)/\ell$. Let

$$\text{MCM}(D) = \min_C m(C),$$

where C ranges over all directed cycles in D . $\text{MCM}(D)$ is called the *minimum cycle mean* of the edge weighted digraph D . The *minimum mean cycle problem* is to determine $\text{MCM}(D)$ for an input edge weighted digraph D . Karp [21] designed an algorithm that solves the minimum mean cycle problem in time $O(nm)$, where n, m are the number of vertices and the number of edges of D , respectively. The following lemma shows that the problem of determining $\gamma_o(w)$ for a given acyclic orientation w is equivalent to a minimum mean cycle problem.

Lemma 11 *Let w be an acyclic orientation of an undirected graph G . Then $\gamma_o(w) = \text{MCM}(G_w)$ for some edge weighted digraph G_w .*

Proof. Let G_w be the following edge weighted digraph: G_w and G have the same vertex set. If $\overrightarrow{xy}$ is an arc of w , then replace the arc $\overrightarrow{xy}$ in w by two arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ having weights $g(\overrightarrow{xy}) = 1$ and $g(\overrightarrow{yx}) = 0$.

Suppose that $C = e_0 e_1 \cdots e_{\ell-1}$ is a directed closed walk in G_w . Let W be the closed walk in G corresponding to C . The weight of C is $g(C) = \sum_{i=0}^{\ell-1} g(e_i) = |W_w^+|$ and hence $m(C) = |W_w^+|/|W|$. Therefore $\text{MCM}(G_w) = \gamma_o(w)$. ■

Theorem 12 *There is an algorithm that determines $\gamma_o(w)$ and an SF-scheduling f of G with $\text{rate}(f) = \gamma_o(w)$ in time $O(nm)$, where n is the number of vertices, and m is the number of edges.*

Proof. By applying Karp's minimum mean cycle algorithm [21] to G_w defined in Lemma 11, we can find $\gamma_o(w)$ in time $O(nm)$. Suppose that $\gamma_o(w) = q/p$ such that p and q are two coprime positive integers. A careful analysis of Karp's algorithm shows that it can be used to determine $p(x, y) = \max\{q|W| - p|W_w^+| : W \text{ is a path from } x \text{ to } y\}$, for all vertices x, y in time $O(nm)$. Now by using the proof of Lemma 6, we obtain an algorithm which finds an SF-scheduling f of G with operating rate q/p in time $O(nm)$. The algorithm is as follows:

Algorithm

Input: Graph G with an acyclic orientation w .

Output: An SF-scheduling f of G with $\text{rate}(f) = \gamma_o(w)$.

Begin

1. Apply Karp's algorithm to determine $\gamma_o(w)$.
2. Fix a vertex $x \in V(G)$.
3. For each $y \in V(G)$, use Karp's algorithm again to find $\tilde{\varphi}(y) = \max\{q|W| - p|W_w^+| : W \text{ is a path from } x \text{ to } y\}$, where p, q are two positive integers with $q/p = \gamma_o(w)$.
4. For each $y \in V(G)$, let $\varphi(y) \equiv \tilde{\varphi}(y) \pmod{p}$ such that $0 \leq \varphi(y) \leq p - 1$.
5. For each positive integer i ,
 let $f(i) = \{v \in V(G) : \varphi(v) \in \{iq - 1, iq - 2, iq - 3, \dots, iq - q\}\}$,
 where all additions and subtractions are modulo p .

End

Then f is an SF-scheduling of G with $\text{rate}(f) = q/p = \gamma_o(w)$. ■

Notice that, in Theorem 12, each $f(i)$ has a simple closed-form expression. The scheduling f constructed by the algorithm above is strongly fair, while the edge reversal scheduling f_w induced by w may not be strongly fair. We also remark that the scheduling f is in general different from the F-scheduling f_w . However, they have the same operating rate.

5 Equivalence of WF-scheduling and fractional coloring

This section considers weakly fair schedulings. We prove the following theorem.

Theorem 13 *For any graph G , $\gamma_w^*(G) = 1/\chi_f(G)$.*

Proof. Suppose that G admits a homomorphism ϕ to Kneser graph $K(p, q)$. Define $f : \{1, 2, 3, \dots\} \rightarrow 2^{V(G)}$ such that $f(i) = \{v \in V(G) : i \in \phi(v)\}$ for $1 \leq i \leq p$ and $f(i) = f(i - p)$ for $i > p$. Then it is easy to verify that f is a WF-scheduling of G with $\text{rate}(f) = q/p$. Therefore $\gamma_w^*(G) \geq q/p$ and hence $\gamma_w^*(G) \geq 1/\chi_f(G)$.

Suppose that f is a WF-scheduling of G . There exists a constant C s.t. $|f(u, i) - f(v, i)| \leq C$ for any $u, v \in V(G)$ and integer $i \geq 1$. For each positive integer i , denote by m_i the value $\min\{|f(v, i)| : v \in V(G)\}$. One can easily construct a homomorphism from G to $K(i, m_i)$ by pruning away $|f(v, i) - m_i|$ vertices from the set $\{j \in \{1, 2, \dots, i\} : v \in f(j)\}$ for each vertex v in G . Thus

$$\frac{1}{\chi_f(G)} \geq \limsup_{i \rightarrow \infty} \frac{m_i}{i} \geq \limsup_{i \rightarrow \infty} \frac{\sum_{j=1}^i |f(j)| - |V(G)|C}{i|V(G)|} = \text{rate}(f).$$

It follows that $\gamma_w^*(G) \leq 1/\chi_f(G)$ which completes the proof. ■

This result shows that weakly fair schedulings could be considerably more efficient than fair schedulings. Indeed, the Kneser graph $K(3k, k)$ has chromatic number $k + 2$, and hence has circular chromatic number $\chi_c(K(3k, k)) > k + 1$ (it was conjectured [20] that $\chi_c(K(p, q)) = \chi(K(p, q))$ for any p, q and the conjecture is partially confirmed [17, 26]). This implies that $\gamma^*(K(3k, k)) < 1/(k + 1)$. On the other hand, $\chi_f(K(3k, k)) = 3$ and hence $\gamma_f^*(K(3k, k)) = 1/3$.

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